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REVIEW

Generalised measures of reliability for multiple outliers

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Abstract The application of the theory of reliability has become a fundamental part of measurement analysis, whether in order to optimise measurement systems so that they are resistant to the influence of outliers or in the post-analysis identification of outliers. However, the current theory of reliability is based on the assumption of a single outlier—an assumption that may not necessarily be the case. This paper extends reliability theory so that it can be applied to multiple outliers through the derivation of appropriate measures of reliability for multiple outliers. The measures of reliability covered include minimal detectable biases, reliability numbers, controllability, and external reliability.

Keywords Multiple outliers · MDB · Reliability numbers · Controllability · External reliability

1 Introduction

Current theory of reliability (Baarda 1967, 1968, 1977; Pope 1975 and so on) is based on the assumption of a single outlier. However, in practice, there could be more than one outlier. For example, if a geodesist considers that 1 in 100 measurements is an outlier, from past experience, and is to carry out a network with 50 measurements. Then there is a 50% probability that the network contains one outlier, a 12% probability of two, a 4% probability of three, and a 2% probability of four or more. Hence, if the probability of experiencing four or more is deemed remote enough to ignore then the geodesist may wish to design a network that is resistant to three outliers. Therefore, measures of reliability for multiple outliers are required.

One part of reliability theory that has been generalised to multiple outliers is the outlier test for non-singular variance covariance matrices (Cook and Weisberg 1982; Förstner 1983; Kok 1984; Belsley et al. 1980; Chatterjee and Hadi 1988; Draper and Smith 1998), singular variance covariance matrices (Wang and Chen 1999), and when the variance factor is unknown (Chen et al. 1987). In addition, these multiple outlier tests have also been shown to be uniformly most powerful (Kargoll 2007; Teunissen 1991). It has also been demonstrated that in the presence of outliers the non-central parameter of the multiple outlier statistic is equivalent to the non-central parameter of the global model statistic (Förstner 1983; Kok 1984; Wang and Chen 1999). Hence, using this property, which is similar to the single outlier case, Kok (1984) generalised the β-method for multiple outliers.

In the case of internal reliability, some attempts have been made to obtain the minimal detectable bias (MDB) vector for multiple outliers. Förstner (1983), Snow (2002), and Wang and Chen (1999) separate the MDB vector into scalar and vector components, and then obtain the scalar component using an assumed vector component. Ryan and Lachapelle (2001) use simulations to obtain the MDB polygon for two outliers. However, explicit formulae to obtain the minimal detectable outlier in a particular measurement are not available.

Consequently, the related measures of reliability, including reliability numbers (Pelzer 1980; Wang and Chen 1994; Chen and Wang 1996; Schaffrin 1997; Ou 1999) and controllability (Pelzer 1980; Förstner 1985), have not been generalised for multiple outliers. Progress has nevertheless been

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made in generalising and applying redundancy numbers for multiple outliers. Förstner (1987) obtained redundancy numbers from the iterative application of the single outlier test, while Schaffrin and Toutenburg (1998) obtain redundancy numbers for the missing values problem. Förstner (1994) and Corthren (2005) also introduce the concept of the redundancy sub-matrix. Components of the redundancy matrix are used by Cross and Price (1985) and Ding and Coleman (1996a,b) to determine the number of outliers and to reject multiple outliers simultaneously. Prószyński (1997, 2000) also uses the redundancy matrix to evaluate the hiding effects of multiple outliers.

External reliability for multiple outliers can be obtained by substitution of the MDB vector into the least squares solution (Förstner 1983; Wang and Chen 1999; Ryan and Lachapelle 2001). However, since numerous MDB vectors are possible for any combination of outliers (Ober 1996; Ryan and Lachapelle 2001; Angus 2006), such a procedure many not yield the largest undetected influence on the parameters. Consequently, Ober (1996) and Angus (2006) utilised the Rayleigh–Ritz Theorem to obtain the maximum external reliability.

The measure of external reliability given by Baarda (1977) is the sum of the weighted external reliability vector, which is also referred to as the sensitivity factor (Förstner 1983, 1985). Förstner (1983) outlined the procedure for obtaining the sensitivity factor for multiple outliers using the Rayleigh–Ritz Theorem.

To obtain a more complete set of measures of internal and external reliability for multiple outliers this paper derives a unique formula for the MDBs in the presences of multiple outliers. Consequently, the controllability and reliability numbers are also obtained. Then, the computation of external reliability for multiple outliers is described.

2 Hypothesis tests and multiple outliers

2.1 The linear model

The Gauss–Markov model is given by,

$$\mathbf{v} = \mathbf{A}\mathbf{x} - \ell;\ \mathrm{E}(\mathbf{v}) = 0 \tag{1}$$

where $\mathbf{v}$ is the residuals vector, $\mathbf{A}$ is the n by t design matrix with rank t, $\mathbf{x}$ is the vector of t parameters solved for, and ℓ is the vector of n measurements. The n by n positive definite variance covariance matrix, which implies full rank, of the measurements Σ is given by,

$$D(\ell) = \Sigma = \sigma_0^2\mathbf{Q} = \sigma_0^2\mathbf{P}^{-1} \tag{2}$$

where σ_0^2 is the a priori variance factor, $\mathbf{Q}$ is the cofactor matrix, and $\mathbf{P}$ is the weight matrix.

2.2 The global model test

The global model test is used to detect discrepancies between the measurements, and the functional and stochastic models. The test is carried out on the a priori and a posteriori variance factors. That is,

$$\begin{aligned} &\mathrm{H}_0 : \mathrm{E}\left\{f\hat{\sigma}_0^2/\sigma_0^2\right\} = f \\ &\mathrm{H}_\mathrm{a} : \mathrm{E}\left\{f\hat{\sigma}_0^2/\sigma_0^2\right\} \neq f \end{aligned} \tag{3}$$

where $\hat{\sigma}_0^2$ is the a posteriori variance factor, and f is the number of redundancies, satisfying,

$$1 \leq f = (n - t) \tag{4}$$

Hence, the global model test statistic can be formulated as,

$$\frac{f\hat{\sigma}_0^2}{\sigma_0^2} = \frac{\mathbf{v}^\mathrm{T}\mathbf{P}\mathbf{v}}{\sigma_0^2} = \frac{\ell^\mathrm{T}\mathbf{P}\mathbf{Q}_\mathbf{v}\mathbf{P}\ell}{\sigma_0^2} \sim \chi^2_{1-\alpha_\mathrm{g}, f} \tag{5}$$

where α_g is the level of significance for the global model test and $\mathbf{Q}_\mathbf{v}$ is the cofactor matrix of the estimated residuals, given by,

$$\mathbf{Q}_\mathbf{v} = \mathbf{P}^{-1} - \mathbf{A}(\mathbf{A}^\mathrm{T}\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^\mathrm{T} \tag{6}$$

If the test fails and the functional and stochastic models are not at fault, it is deduced that the test fails because of the presence of one or more outliers in the measurements. The statistic then follows a non-central chi-squared distribution, with non-central parameter given by (Baarda 1967; Teunissen 2000, 2006),

$$\lambda = \frac{\mathbf{z}^\mathrm{T}\mathbf{H}^\mathrm{T}\mathbf{P}\mathbf{Q}_\mathbf{v}\mathbf{P}\mathbf{H}\mathbf{z}}{\sigma_0^2} \tag{7}$$

where $\mathbf{z}$ is the true vector of outliers, and $\mathbf{H}$ corresponds to the true outlier vector.

2.3 The outlier test

The outlier test can be used to identify the outlying measurements. This is provided that the number of outliers considered, θ, satisfies the inequality (Hewitson et al. 2004),

$$1 \leq \theta \leq f = (n - t) \tag{8}$$

The outlier test can be derived from the mean shift model (Cook and Weisberg 1982; Kok 1984),

$$\mathbf{v} = \begin{bmatrix} \mathbf{A} & \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} - \ell; \quad \mathrm{E}(\ell) = 0 \tag{9}$$

where $\mathbf{z}$ is a vector of θ outliers solved for, and $\mathbf{H}$ is an n by θ matrix, with rank θ, containing zeros with a one in each column corresponding to an outlier. Then using partitioned matrixes to solve Eq. (9) for the outlier vector yields,

$$\hat{\mathbf{z}} = (\mathbf{H}^\mathrm{T}\mathbf{P}\mathbf{Q}_\mathbf{v}\mathbf{P}\mathbf{H})^{-1}\mathbf{H}^\mathrm{T}\mathbf{P}\mathbf{Q}_\mathbf{v}\mathbf{P}\ell \tag{10}$$

with a variance covariance matrix of,

$$\Sigma_{\hat{\mathbf{z}}} = \sigma_0^2(\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathrm{v}}\mathbf{P}\mathbf{H})^{-1} \tag{11}$$

Therefore the outlier statistic (Förstner 1983; Kok 1984; Wang and Chen 1999),

$$w^2 = \hat{\mathbf{z}}^{\mathrm{T}}\Sigma_{\hat{\mathbf{z}}}^{-1}\hat{\mathbf{z}} = \frac{\ell^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathrm{v}}\mathbf{P}\mathbf{H}(\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathrm{v}}\mathbf{P}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathrm{v}}\mathbf{P}\ell}{\sigma_0^2} \sim \chi^2_{1-\alpha_{w^2},\,\theta} \tag{12}$$

can be formed, for a given $\mathbf{H}$ matrix, where $\alpha_{w^2}$is the level of significance for the outlier test. Since there are $\binom{n}{\theta}$combinations of the $\mathbf{H}$ matrix that can be formed for θ outliers, then there are also $\binom{n}{\theta}$ w^2 statistics. The hypothesis that is then tested for each w^2 statistic is,

$$\begin{aligned} \mathrm{H}_0 &: \mathrm{E}\{\hat{\mathbf{z}}\} = 0 \\ \mathrm{H}_{\mathrm{a}} &: \mathrm{E}\{\hat{\mathbf{z}}\} \neq 0 \end{aligned} \tag{13}$$

If one of the outlier test statistics fails, it is concluded that one or more outliers are contained within the measurements. If identification is possible, then the largest w^2 statistic is expected to correspond to the true outlier vector $\mathbf{z}$. Since the statistic becomes a non-central chi-square distribution with non-central parameter given by (Baarda 1968; Förstner 1983; Teunissen 2000, 2006; Wang and Chen 1999),

$$\lambda = \frac{\mathbf{z}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathrm{v}}\mathbf{P}\mathbf{H}\mathbf{z}}{\sigma_0^2} \tag{14}$$

Hence, the measurements that contain the outliers can then be identified from the $\mathbf{H}$ matrix corresponding to the largest w^2 statistic.

However, since in practice, the true number of outliers is unknown and all that can be obtained is an estimate of the maximum number of outliers to be reasonably encountered. Then the procedure is to apply the outlier test in Eq. (12) for θ equal to one and determine the most likely suspect based on the assumption of a single outlier. Then the outlier test in Eq. (12) is applied for θ equal to two and the most likely suspects based on the assumption of two outliers are determined. This process is then continued until θ is equal to the maximum number of outliers to be reasonably considered. Hence, from the illustration in Sect. 1 the outlier test in Eq. (12) would be carried out for θ equal to one, two and three. The suspect measurements based on the varying number of outliers are then used as a starting point for further investigations (Baarda 1968; Pope 1975).

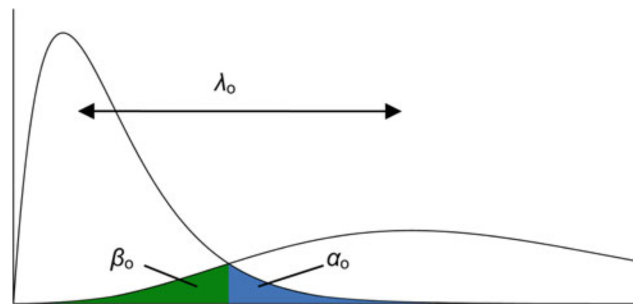

Fig. 1 Chi-square distributions and the non-central parameter

3 Internal reliability

Despite the use of rigorous statistical testing procedures, unfortunately, the presence of one or more outliers may go undetected using the global model test or the outlier test. Consequently, it is desirable to have some knowledge of the magnitude of an outlier vector that can be present, for a given set of Type I and Type II error probabilities. That is, after selecting Type I error α_0, and Type II error β_0, probabilities the non-central parameter, λ_0, can be obtained by iteratively solving,

$$\chi^2_{1-\alpha_0,d} = \chi^2_{\beta_0,d,\lambda_0} \tag{15}$$

where d is the degrees of freedom. This process is also schematically shown in Fig. 1. Then using the specified non-central parameter, λ_0, in Eq. (7) or (14), the corresponding outlier vector $\mathbf{z}_0$ can be obtained that is just detectable for the probabilities α_0 and β_0.

Such a process can be carried out for the global model test to obtain the non-central parameter λ_{g} as a function of,

$$\lambda_{\mathrm{g}} = \lambda\left(\alpha_{\mathrm{g},}\beta_{\mathrm{g}},\, f\right) \tag{16}$$

and then the corresponding internal reliability vector $\mathbf{z}_{\mathrm{g}}$ can be obtained from Eq. (7). Likewise, for the outlier test the non-central parameter λ_{w^2} can be obtained as a function of,

$$\lambda_{w^2} = \lambda(\alpha_{w^2,}\beta_{w^2},\theta) \tag{17}$$

and the corresponding outlier vector $\mathbf{z}_{w^2}$ can be obtained from Eq. (14).

It should be noted that in the special case when λ_{g} is equal to λ_{w^2}, the outlier vectors $\mathbf{z}_{\mathrm{g}}$ and $\mathbf{z}_{w^2}$ are equivalent. Consequently, if the probabilities are appropriately selected then the outlier vectors $\mathbf{z}_{\mathrm{g}}$ and $\mathbf{z}_{w^2}$ can be made equivalent. One such method is the β-Method (Baarda 1968; Kok 1984).

However, regardless of the probabilities and the test utilised from this point forth the notation λ_0 for the non-central parameter and $\mathbf{z}_0$ for the corresponding outlier vector will be adopted. This is because the proceeding sections are equally applicable for the global model test and the outlier test. Hence λ_0 and $\mathbf{z}_0$ can be simply replaced by the corresponding λ_{g}

and $\mathbf{z}_g$ for the global model test or λ_{w^2} and $\mathbf{z}_{w^2}$ for the outlier test.

3.1 A single outlier

If there is only a single outlier, that is θ equals one, then the outlier vector reduces to a scalar z. Therefore, for a given λ_0, a unique solution can be obtained from Eq. (7) or (14) for the MDB in the ith observation as (Baarda 1967, 1968; Teunissen 2000, 2006),

$$z_{0i} = \sqrt{\frac{\lambda_0 \sigma_0^2}{\mathbf{h}_i^{\mathrm{T}} \mathbf{P} \mathbf{Q}_{\mathbf{v}} \mathbf{P} \mathbf{h}_i}} \tag{18}$$

where $\mathbf{H}$ has reduced to the single column vector $\mathbf{h}$. Since there are now $\binom{n}{1}$ combinations of the vector $\mathbf{h}$ there is also an equal number of z_{0i}.

3.2 Multiple outliers

If there is more than a single outlier then a unique solution cannot be obtained for the MDB vector from Eq. (7) or (14) for a given λ_0. It is due to this reason that Ryan and Lachapelle (2001) simulate the MDB polygon for two outliers.

If, however, the MDB vector is split into a unit vector component $\mathbf{z}_\mathbf{u}$, and a scalar component z_s, then by assuming a unit vector component, that is a ratio of outliers, the scalar component can be obtained from Eq. (7) or (14) as (Förstner 1983; Snow 2002; Wang and Chen 1999),

$$z_s = \sqrt{\frac{\lambda_0 \sigma_0^2}{\mathbf{z}_\mathbf{u}^{\mathrm{T}} \mathbf{H}^{\mathrm{T}} \mathbf{P} \mathbf{Q}_{\mathbf{v}} \mathbf{P} \mathbf{H} \mathbf{z}_\mathbf{u}}} \tag{19}$$

Hence, the corresponding MDB vector is,

$$\mathbf{z}_0 = z_s \mathbf{z}_\mathbf{u} = \sqrt{\frac{\lambda_0 \sigma_0^2}{\mathbf{z}_\mathbf{u}^{\mathrm{T}} \mathbf{H}^{\mathrm{T}} \mathbf{P} \mathbf{Q}_{\mathbf{v}} \mathbf{P} \mathbf{H} \mathbf{z}_\mathbf{u}}}\, \mathbf{z}_\mathbf{u} \tag{20}$$

that can be evaluated for all $\binom{n}{\theta}$ combinations of the $\mathbf{H}$ matrix.

This procedure will result in a MDB vector for a particular ratio of outliers. However, with outliers being random in nature, consequently, the ratio of outliers is unknown. Then it would be prudent to avoid the selection of an assumed ratio of outliers $\mathbf{z}_\mathbf{u}$, that is unlikely to yield the maximum MDB in the ith observation even when all $\binom{n}{\theta}$ combinations of the $\mathbf{H}$ matrix are considered. Therefore, a procedure that obtains the maximum MDB in the ith observation when θ outliers are considered is desired.

3.2.1 Maximum MDB for θ outliers

One procedure for obtaining the maximum MDB in the ith observation when θ outliers are considered is via the Rayleigh–Ritz Theorem (Appendix A). To apply the Rayleigh–Ritz Theorem it is convenient to consider the optimisation problem as maximising $\mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x}$ subject to the constraint of $\mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x}$ being equal to one. In this case $\mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x}$ is obtained from Eq. (7) or (14) as,

$$\mathbf{z}_0^{\mathrm{T}} \left(\frac{\mathbf{H}^{\mathrm{T}} \mathbf{P} \mathbf{Q}_{\mathbf{v}} \mathbf{P} \mathbf{H}}{\lambda_0 \sigma_0^2} \right) \mathbf{z}_0 = 1 \tag{21}$$

where $\mathbf{B}$ satisfies the condition of a symmetrical positive definite matrix. Provided that Eq. (8) is satisfied irrespective of whether the MDB is computed for the Global Model Test or the outlier test. The $\mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x}$ value is then formulated for the ith observation as,

$$\mathbf{z}_0^{\mathrm{T}} \mathbf{c}_{\theta i}^{\mathrm{T}} \mathbf{c}_{\theta i} \mathbf{z}_0 \tag{22}$$

where $\mathbf{c}_{\theta i}$ is a one by θ vector of zeros with a one corresponding to the ith outlier in $\mathbf{z}_0$. This results in $\mathbf{c}_{\theta i}^{\mathrm{T}} \mathbf{c}_{\theta i}$ forming a θ by θ matrix of zeros with a one in the diagonal element corresponding to the ith measurement. Hence, Eq. (22) reduces to $(z_{0i}^{\theta})^2$, being the square of the MDB in the ith observation when θ outliers are considered. Therefore, the maximum z_{0i}^{θ} can be obtained via,

$$\lambda_{\mathrm{Min}} \le \frac{\mathbf{z}_0^{\mathrm{T}} \mathbf{c}_{\theta i}^{\mathrm{T}} \mathbf{c}_{\theta i} \mathbf{z}_0}{\mathbf{z}_0^{\mathrm{T}} \left(\frac{\mathbf{H}^{\mathrm{T}} \mathbf{P} \mathbf{Q}_{\mathbf{v}} \mathbf{P} \mathbf{H}}{\lambda_0 \sigma_0^2} \right) \mathbf{z}_0} \le \lambda_{\mathrm{Max}} \tag{23}$$

where the eigenvalues and eigenvectors are obtained from,

$$\left(\lambda_0 \sigma_0^2 \left(\mathbf{H}^{\mathrm{T}} \mathbf{P} \mathbf{Q}_{\mathbf{v}} \mathbf{P} \mathbf{H} \right)^{-1} \mathbf{c}_{\theta i}^{\mathrm{T}} \mathbf{c}_{\theta i} \right) \mathbf{u} = \lambda \mathbf{u} \tag{24}$$

Hence the maximum z_{0i}^{θ} is obtained from the maximum eigenvalue by,

$$z_{0i}^{\theta} = \sqrt{\lambda_{\mathrm{Max}}} \tag{25}$$

with the corresponding outlier vector obtained from,

$$\mathbf{z}_{0\mathrm{Max}} = \mathbf{u}_{\mathrm{Max}} \tag{26}$$

where $\mathbf{u}_{\mathrm{Max}}$ is the eigenvector corresponding to the maximum eigenvalue. In addition, the ith value in $\mathbf{z}_{0\mathrm{Max}}$ is equivalent to that from Eq. (25).

Alternatively, the eigenvalues and eigenvectors can be obtained from,

$$\left(\left(\mathbf{U}^{\mathrm{T}} \right)^{-1} \mathbf{c}_{\theta i}^{\mathrm{T}} \mathbf{c}_{\theta i} \mathbf{U}^{-1} \right) \mathbf{u}_* = \lambda \mathbf{u}_* \tag{27}$$

where $\mathbf{U}$ is the upper triangle from the Cholesky decomposition of,

$$\frac{\mathbf{H}^{\mathrm{T}} \mathbf{P} \mathbf{Q}_{\mathbf{v}} \mathbf{P} \mathbf{H}}{\lambda_0 \sigma_0^2} = \mathbf{U}^{\mathrm{T}} \mathbf{U} \tag{28}$$

Hence the maximum z_{0i}^{θ} is given by,

$$z_{0i}^{\theta} = \sqrt{\lambda_{\mathrm{Max}}} \tag{29}$$

with the corresponding outlier vector now obtained from,

$$\mathbf{z}_{0\mathrm{Max}} = \mathbf{U}^{-1}\mathbf{u}_{*\mathrm{Max}} \tag{30}$$

where $\mathbf{u}_{*\mathrm{Max}}$ is the eigenvector corresponding to the maximum eigenvalue.

The above procedure, while obtaining the maximum MDB in the ith observation for θ outliers, does not provide great insight into the factors affecting internal reliability. However, if the procedure using Cholesky decomposition is carried out with the $\mathbf{H}$ matrix partitioned as,

$$\mathbf{H} = \left[\mathbf{H}_j \;\; \mathbf{h}_i\right] = \left[\mathbf{H}_j \;\; \mathbf{H}\mathbf{c}_{\theta i}^{\mathrm{T}}\right] \tag{31}$$

then in Eq. (28),

$$\frac{\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}}{\lambda_0\sigma_0^2} = \frac{1}{\lambda_0\sigma_0^2}\begin{bmatrix} \mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}_j & \mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i \\ \mathbf{h}_i^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}_j & \mathbf{h}_i^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i \end{bmatrix} \tag{32}$$

and denoting as $\mathbf{G}$,

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{jj} & \mathbf{g}_{ji} \\ \mathbf{g}_{ji}^{\mathrm{T}} & g_{ii} \end{bmatrix} = \frac{1}{\lambda_0\sigma_0^2}\begin{bmatrix} \mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}_j & \mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i \\ \mathbf{h}_i^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}_j & \mathbf{h}_i^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i \end{bmatrix} \tag{33}$$

the Cholesky decomposition of $\mathbf{G}$ is,

$$\mathbf{G} = \mathbf{U}^{\mathrm{T}}\mathbf{U} = \begin{bmatrix} \mathbf{U}_{jj}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{g}_{ji}^{\mathrm{T}}\mathbf{U}_{jj}^{-1} & \sqrt{g_{ii} - \mathbf{g}_{ji}^{\mathrm{T}}\mathbf{G}_{jj}^{-1}\mathbf{g}_{ji}} \end{bmatrix} \times \begin{bmatrix} \mathbf{U}_{jj} & \left(\mathbf{U}_{jj}^{\mathrm{T}}\right)^{-1}\mathbf{g}_{ji} \\ \mathbf{0} & \sqrt{g_{ii} - \mathbf{g}_{ji}^{\mathrm{T}}\mathbf{G}_{jj}^{-1}\mathbf{g}_{ji}} \end{bmatrix} \tag{34}$$

where $\mathbf{U}_{jj}^{\mathrm{T}}\mathbf{U}_{jj}$ is the Cholesky decomposition of $\mathbf{G}_{jj}$. Hence, the inverse of $\mathbf{U}$ can also be obtained as,

$$\mathbf{U}^{-1} = \begin{bmatrix} \mathbf{U}_{jj}^{-1} & -\mathbf{G}_{jj}^{-1}\mathbf{g}_{ji}\Big/\sqrt{g_{ii} - \mathbf{g}_{ji}^{\mathrm{T}}\mathbf{G}_{jj}^{-1}\mathbf{g}_{ji}} \\ \mathbf{0} & 1\Big/\sqrt{g_{ii} - \mathbf{g}_{ji}^{\mathrm{T}}\mathbf{G}_{jj}^{-1}\mathbf{g}_{ji}} \end{bmatrix} \tag{35}$$

Therefore, if $\mathbf{c}_{\theta i}$ is $[\mathbf{0}\ 1]$, then in Eq. (27),

$$(\mathbf{U}^{-1})^{\mathrm{T}}\mathbf{c}_{\theta i}^{\mathrm{T}}\mathbf{c}_{\theta i}\mathbf{U}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1/(g_{ii} - \mathbf{g}_{ji}^{\mathrm{T}}\mathbf{G}_{jj}^{-1}\mathbf{g}_{ji}) \end{bmatrix} \tag{36}$$

with $\theta - 1$ eigenvalues equal to zero and the maximum eigenvalue given by,

$$\lambda_{\mathrm{Max}} = \frac{1}{g_{ii} - \mathbf{g}_{ji}^{\mathrm{T}}\mathbf{G}_{jj}^{-1}\mathbf{g}_{ji}} \tag{37}$$

then the unique formula for the maximum z_{0i}^{θ} can be obtained as,

$$z_{0i}^{\theta} = \sqrt{\frac{\lambda_0\sigma_0^2}{\mathbf{h}_i^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i - \mathbf{h}_i^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}_j\left(\mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}_j\right)^{-1}\mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i}} \tag{38}$$

This formula can be further simplified, by identifying that the variance covariance matrix of $\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{v}$ is,

$$\Sigma_{\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{v}} = \sigma_0^2\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H} = \sigma_0^2\begin{bmatrix} \mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}_j & \mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i \\ \mathbf{h}_i^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}_j & \mathbf{h}_i^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i \end{bmatrix} \tag{39}$$

which is related to $\Sigma_{\hat{\mathbf{z}}}$ by $\sigma_0^4\Sigma_{\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{v}}^{-1}$. Hence, the ith multiple correlation coefficient is given by (Anderson 1984),

$$\wp_{\mathrm{H^TPv}\ \mathrm{i}} = \sqrt{\frac{\mathbf{h}_i^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}_j\left(\mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}_j\right)^{-1}\mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i}{\mathbf{h}_i^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i}} \tag{40}$$

where there are $\binom{n-1}{\theta-1}$ combinations of the $\mathbf{H}_j$ matrix associated with the ith measurement. It is also noted that in the case of two outliers the multiple correlation coefficient is equivalent to the absolute value of the correlation coefficients between two single outlier statistics (Förstner 1983). The unique formula for z_{0i}^{θ} in Eq. (38) then becomes,

$$z_{0i}^{\theta} = \sqrt{\frac{\lambda_0\sigma_0^2}{\mathbf{h}_i^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i(1 - \wp_{\mathrm{H^TPv}\ \mathrm{i}}^2)}} \tag{41}$$

where there are now $\binom{n-1}{\theta-1}$ values associated with the ith measurement.

If the MDB for a single outlier in Eq. (18) is then substituted into Eq. (41),

$$z_{0i}^{\theta} = \frac{z_{0i}}{\sqrt{1 - \wp_{\mathrm{H^TPv}\ \mathrm{i}}^2}} \tag{42}$$

and noting that the bounds of $\wp_{\mathrm{H^TPv}\ \mathrm{i}}$ are,

$$0 \le \wp_{\mathrm{H^TPv}\ \mathrm{i}} \le 1 \tag{43}$$

it can be then concluded that the MDB for θ outliers in the ith measurement is greater than or equal to the corresponding MDB for a single outlier.

However, regardless of the method chosen, the full evaluation of the minimal detectable outlier in a particular observation requires the calculation of $n\binom{n-1}{\theta-1}$ combinations, that is $\theta\binom{n}{\theta}$ combinations.

3.3 Controllability

Controllability is a measure of internal reliability that is derived from the Minimal Detectable Biases. Controllability for the ith measurement C_{0i} is given by (Pelzer 1980; Förstner 1985),

$$z_{0i} = C_{0i}\sigma_i \tag{44}$$

where σ_i is the standard deviation of the ith measurement.

Therefore, in the single outlier case controllability can be obtained by multiplying Eq. (18) by σ_i/σ_i for the ith measurement to give,

$$z_{0i} = \frac{\sigma_i}{\sigma_i}\sqrt{\frac{\lambda_0\sigma_0^2}{\mathbf{h}_i^{\mathrm{T}}\mathbf{PQ_vPh}_i}} = \sqrt{\frac{\lambda_0}{\mathbf{h}_i^{\mathrm{T}}\mathbf{Qh}_i\mathbf{h}_i^{\mathrm{T}}\mathbf{PQ_vPh}_i}}\sigma_i \tag{45}$$

where the controllability is obtained as (Pelzer 1980; Wang and Chen 1994; Chen and Wang 1996);

$$C_{0i} = \sqrt{\frac{\lambda_0}{\mathbf{h}_i^{\mathrm{T}}\mathbf{Qh}_i\mathbf{h}_i^{\mathrm{T}}\mathbf{PQ_vPh}_i}} \tag{46}$$

If multiple outliers are considered, then from Eq. (41) it can be deduced (similarly to the single outlier case) that the controllability of the ith measurement for θ outliers C_{0i}^{θ} is,

$$C_{0i}^{\theta} = \sqrt{\frac{\lambda_0}{\mathbf{h}_i^{\mathrm{T}}\mathbf{Qh}_i\mathbf{h}_i^{\mathrm{T}}\mathbf{PQ_vPh}_i\left(1-\wp^2_{\mathrm{H^TPv}\ i}\right)}} = \frac{C_{0i}}{\sqrt{1-\wp^2_{\mathrm{H^TPv}\ i}}} \tag{47}$$

that is greater than or equal to C_{0i} for a single outlier. It is also noted that there are now $\binom{n-1}{\theta-1}$ controllability values associated with each measurement.

3.4 Reliability numbers

Reliability numbers are derived from controllability, and remove the effect of the non-central parameter λ_0.

For the single outlier case the reliability numbers are given as (Pelzer 1980; Wang and Chen 1994; Chen and Wang 1996),

$$\bar{r}_i = \mathbf{h}_i^{\mathrm{T}}\mathbf{Qh}_i\mathbf{h}_i^{\mathrm{T}}\mathbf{PQ_vPh}_i \tag{48}$$

with the bounds of,

$$0 \le \bar{r}_i \le \mathbf{h}_i^{\mathrm{T}}\mathbf{Qh}_i\mathbf{h}_i^{\mathrm{T}}\mathbf{Ph}_i \tag{49}$$

If the measurements are uncorrelated then the reliability numbers are equivalent to the redundancy numbers (Förstner 1979),

$$r_i = \mathbf{h}_i^{\mathrm{T}}\mathbf{Q_vPh}_i \tag{50}$$

that have the bounds of,

$$0 \le r_i \le 1 \tag{51}$$

and sum to f.

Similar to the single outlier case, reliability numbers can also be obtained for multiple outliers. The generalisation of reliability numbers, defined by Pelzer (1980) and Wang and Chen (1994), to θ outliers is,

$$\begin{aligned}\bar{r}_i^{\theta} &= \mathbf{h}_i^{\mathrm{T}}\mathbf{Qh}_i\mathbf{h}_i^{\mathrm{T}}\mathbf{PQ_vPh}_i\left(1-\wp^2_{\mathrm{H^TPv}\ i}\right)\\ &= \bar{r}_i\left(1-\wp^2_{\mathrm{H^TPv}\ i}\right)\end{aligned} \tag{52}$$

with the bounds of,

$$0 \le \bar{r}_i^{\theta} \le \mathbf{h}_i^{\mathrm{T}}\mathbf{Qh}_i\mathbf{h}_i^{\mathrm{T}}\mathbf{Ph}_i\left(1-\wp^2_{\mathrm{H^TPv}i}\right) \tag{53}$$

If the measurements are uncorrelated then it can be shown that the reliability numbers for multiple outliers are also equivalent to the redundancy numbers for multiple outliers, given by (Förstner 1987),

$$r_i^{\theta} = \mathbf{h}_i^{\mathrm{T}}\mathbf{Q_vPh}_i - \mathbf{h}_i^{\mathrm{T}}\mathbf{Q_vPH}_j\left(\mathbf{H}_j^{\mathrm{T}}\mathbf{PQ_vPH}_j\right)^{-1}\mathbf{H}_j^{\mathrm{T}}\mathbf{PQ_vPh}_i \tag{54}$$

that have the bounds of,

$$0 \le r_i^{\theta} \le 1 \tag{55}$$

In addition, Förstner (1987) demonstrated that the summation of the redundancy numbers for a given $\mathbf{H}_j$ is $f-\theta+1$.

From an inspection of the reliability numbers for multiple outliers, it can be concluded that it is ideal to have large diagonal elements of the $\mathbf{PQ_VP}$ matrix and all off-diagonal elements equal to zero.

4 External reliability

External reliability is the effect of undetected outliers on the estimated parameters.

4.1 A single outlier

In the single outlier case, external reliability is obtained by substituting the unique solution for the MDB in Eq. (18), into the least squares solution, to give (Baarda 1968),

$$\mathbf{y}_{0i} = (\mathbf{A}^{\mathrm{T}}\mathbf{PA})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{Ph}_i z_{0i} \tag{56}$$

where $\mathbf{y}_{0i}$ is the external reliability vector for the MDB in the ith measurement.

4.2 Multiple outliers

For multiple outliers, the external reliability can be obtained in a similar manner to the single outlier case by substitution of the MDB vector into the least squares solution, as (Förstner 1983; Ryan and Lachapelle 2001; Wang and Chen 1999),

$$\mathbf{y}_0 = (\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{H}\mathbf{z}_0. \tag{57}$$

The MDB vector could then be obtain from Eq. (20) for an assumed ratio of outliers, and hence external reliability becomes (Förstner 1983; Wang and Chen 1999),

$$\mathbf{y}_0 = (\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{H}\mathbf{z}_{\mathbf{u}}\sqrt{\frac{\lambda_0\sigma_0^2}{\mathbf{z}_{\mathbf{u}}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}\mathbf{z}_{\mathbf{u}}}} \tag{58}$$

Alternatively, the MDB vector from Eq. (26) or (30) could also be used. In this case a unique formula for external reliability can also be derived, by firstly obtaining the outlier vector. From Eq. (36), the eigenvector corresponding with λ_{Max}, $\mathbf{u}_{*\mathrm{Max}}$, can be obtained as $[\mathbf{0}\ 1]^{\mathrm{T}}$. Hence the MDB vector $\mathbf{z}_{0\mathrm{Max}}$ via Eq. (30) is,

$$\mathbf{z}_{0\mathrm{Max}} = \begin{bmatrix} -\mathbf{G}_{jj}^{-1}\mathbf{g}_{ji} \Big/ \sqrt{g_{ii} - \mathbf{g}_{ji}^{\mathrm{T}}\mathbf{G}_{jj}^{-1}\mathbf{g}_{ji}} \\ 1 \Big/ \sqrt{g_{ii} - \mathbf{g}_{ji}^{\mathrm{T}}\mathbf{G}_{jj}^{-1}\mathbf{g}_{ji}} \end{bmatrix} \tag{59}$$

which can also be written as,

$$\mathbf{z}_{0\mathrm{Max}} = \begin{bmatrix} -\left(\mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}_j\right)^{-1}\mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i\sqrt{\frac{\lambda_0\sigma_0^2}{\mathbf{h}_i^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i\left(1-\wp^2_{\mathrm{H^TPv}\ i}\right)}} \\ \sqrt{\frac{\lambda_0\sigma_0^2}{\mathbf{h}_i^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i\left(1-\wp^2_{\mathrm{H^TPv}\ i}\right)}} \end{bmatrix} \tag{60}$$

or in terms of z_{0i}^{θ} as,

$$\mathbf{z}_{0\mathrm{Max}} = \begin{bmatrix} -\left(\mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}_j\right)^{-1}\mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i z_{0i}^{\theta} \\ z_{0i}^{\theta} \end{bmatrix} \tag{61}$$

Therefore substituting Eq. (61) into Eq. (57) yields the external reliability vector,

$$\mathbf{y}_0 = \Big(\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{h}_i - \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1} \times \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{H}_j\left(\mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}_j\right)^{-1}\mathbf{H}_j^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{h}_i\Big) z_{0i}^{\theta} \tag{62}$$

However, when multiple outliers exist, the MDB vectors obtained from internal reliability are only some of the numerous outlier vectors satisfying Eq. (21), even when all combinations of the **H** matrix are considered. Consequently, the outlier vectors obtained from internal reliability may not contain the outlier vector that maximises external reliability. Therefore, the vector of outliers $\mathbf{z}_0$ desired is the one that maximises external reliability for a particular parameter.

4.2.1 Maximum external reliability for θ outliers

The maximum effect of undetected outliers on the *k*th parameter can be obtained similarly via the Rayleigh–Ritz Theorem. In this, case the constraint of $\mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x}$ remains unchanged to that in Eq. (21). However, since it is desired to maximise the *k*th external reliability parameter y_{0k}^{θ} when θ outliers are considered, then $\mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x}$ is formulated as,

$$\mathbf{z}_0^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A})^{-1}\mathbf{c}_t^{\mathrm{T}}\mathbf{c}_t(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{H}\mathbf{z}_0 \tag{63}$$

where $\mathbf{c}_t$ is a one by t vector of zeros with a one corresponding to the *k*th parameter to be maximised. Hence, Eq. (63) reduces to $(y_{0k}^{\theta})^2$, which is to be maximised. Therefore, the maximum y_{0k}^{θ} can be obtained via (Ober 1996; Angus 2006),

$$\lambda_{\mathrm{Min}} \le \frac{\mathbf{z}_0^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{c}_t^{\mathrm{T}}\mathbf{c}_t\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{H}\mathbf{z}_0}{\mathbf{z}_0^{\mathrm{T}}\left(\frac{\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}}{\lambda_0\sigma_0^2}\right)\mathbf{z}_0} \le \lambda_{\mathrm{Max}} \tag{64}$$

in which the eigenvalues are given by,

$$\Big(\lambda_0\sigma_0^2\left(\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{Q}_{\mathbf{v}}\mathbf{P}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{P}\mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{c}_t^{\mathrm{T}}\mathbf{c}_t \times \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{H}\Big)\mathbf{u} = \lambda\mathbf{u} \tag{65}$$

and hence the maximum y_{0k}^{θ} is,

$$y_{0k}^{\theta} = \sqrt{\lambda_{\mathrm{Max}}} \tag{66}$$

It is also noted that the corresponding outlier vector can be obtained from,

$$\mathbf{z}_{0\mathrm{Max}} = \mathbf{u}_{\mathrm{Max}} \tag{67}$$

and substituted into Eq. (57) with the appropriate **H** matrix to yield the maximum y_{0k}^{θ}. It should also be emphasised that $\mathbf{z}_{0\mathrm{Max}}$ from Eq. (67) is different to that obtained from internal reliability when the *i*th observation is maximised for θ outliers, in Eqs. (26), (30), and (61). Hence the reason for Eq. (62) being unsuitable for obtaining the maximum y_{0k}^{θ}. It is due to these reasons that Ober (1996) and Angus (2006) only demonstrated external reliability for multiple outliers and not internal reliability as given in Sect. 3.

The full evaluation of external reliability for the *k*th parameter involves the evaluation of all $\binom{n}{\theta}$ combinations of **H**.

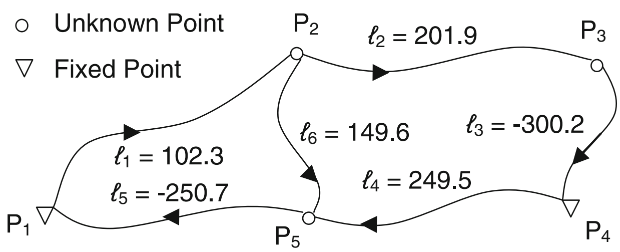

Fig. 2 Levelling network (units meters)

5 Example

As an example, consider the levelling network displayed in Fig. 2 where the control points are both at 1000 m, and the variance covariance matrix of the measurements is given by,

$$\boldsymbol{\Sigma} = \begin{bmatrix} 5.5 & 3.7 & 0.3 & -3.2 & -0.5 & 0.1 \\ 3.7 & 3.9 & 0.0 & -0.8 & -0.6 & -0.7 \\ 0.3 & 0.0 & 0.8 & -1.4 & 0.1 & 0.8 \\ -3.2 & -0.8 & -1.4 & 5.4 & -0.3 & -2.1 \\ -0.5 & -0.6 & 0.1 & -0.3 & 0.2 & 0.3 \\ 0.1 & -0.7 & 0.8 & -2.1 & 0.3 & 1.4 \end{bmatrix} \tag{68}$$

If it is assumed that, there is at most one single outlier within the network. Then the reliability values of the MDBs, reliability numbers and controllability values can be obtained from Eqs. (18), (48) and (46) respectively. Therefore, for a λ_0 of 17.07, the internal reliability values can be obtained as shown in Table 1. The external reliability values for a single outlier can also be obtained from Eq. (56), and the results are displayed in Table 2.

If the observations in Fig. 2 were observed then it can be verified that all of the outlier test statistics in Eq. (12), based on θ being equal to one, pass at the 0.1% significance level. In addition, if an outlier is added to observation 1 of 2.5 m then the outlier statistics shown in Table 1 result. However, all of the outlier statistics pass at the 0.1% significance level, since the critical value is 10.83. The reason for this is that the

Table 1 Internal reliability and outlier testing for one outlier in levelling network

i	σ_i (m)	MDB$_i$ (m)	C_{0i}	$\bar{r}_i$	$w^{2\,a}$	$w^{2\,b}$
1	2.35	2.98	1.27	10.58	8.04	17.82
2	1.97	10.35	5.24	0.62	0.10	0.79
3	0.89	10.35	11.57	0.13	0.10	0.79
4	2.32	2.60	1.12	13.68	6.77	15.47
5	0.45	1.32	2.96	1.95	6.88	15.91
6	1.18	2.59	2.19	3.56	6.47	15.17

[a] One outlier of 2.5 m in observation 1
[b] One outlier of 3.5 m in observation 1

Table 2 External reliability for one outlier in levelling network example

i	y_{02} (m)	y_{03} (m)	y_{05} (m)
1	0.11	1.26	0.05
2	4.01	0.10	1.41
3	4.01	10.25	1.41
4	1.04	1.90	0.06
5	1.29	1.54	1.15
6	1.49	1.12	0.40
Max	4.01	10.25	1.41

Table 3 Multiple correlation coefficients in levelling network example

$\wp_{H^T Pv\, i}$	j					
	1	2	3	4	5	6
i						
1	1	0.41	0.41	0.96	0.98	0.97
2	0.41	1	1.00	0.36	0.50	0.61
3	0.41	1.00	1	0.36	0.50	0.61
4	0.96	0.36	0.36	1	0.98	0.93
5	0.98	0.50	0.50	0.98	1	0.98
6	0.97	0.61	0.61	0.93	0.98	1

MDB of observation 1, in Table 1, is 2.98 m, which is larger than the outlier of 2.5m. However, if the outlier is changed to 3.5 m, which is larger than the MDB, then observation 1 is detected as shown in Table 1.

If reliability is now considered for two outliers then the MDBs, reliability numbers and controllability values can be obtained from Eqs. (42), (52) and (47) given the multiple correlation coefficients in Table 3.

Therefore, the maximum internal reliability values for each measurement when two outliers are considered can be computed as shown in Table 4.

From Table 4, it can be seen that all of the MDBs and the controllability numbers are greater than the single outlier values in Table 1, while the reliability numbers are also smaller. This is particularly so for measurements 2 and 3, when both are considered outliers, as there is no reliability, hence explaining the high multiple correlation coefficients of 1.00 in Table 3.

External reliability for two outliers can be obtained from Eq. (66), and the maximum values for each parameter are shown in Table 5. It can be seen that the external reliability values are considerably increased compared with the single outlier case. This is particularly so for P_3 when measurements 2 and 3 are considered as outliers. Hence, considering two outliers results in lower levels of external reliability.

If the observations in Fig. 2 were observed then it can be verified that all of the outlier test statistics in Eq. (12), based

Table 4 Internal reliability for two outliers in levelling network example

i	j	MDB_i	C_{0i}^{θ}	$\bar{r}_i^{\theta}$	i	j	MDB_i	C_{0i}^{θ}	$\bar{r}_i^{\theta}$
1	2	3.27	1.40	8.76	4	1	9.16	3.94	1.10
1	3	3.27	1.40	8.76	4	2	2.79	1.20	11.87
1	4	10.52	4.48	0.85	4	3	2.79	1.20	11.87
1	5	17.20	7.34	0.32	4	5	13.44	5.78	0.51
1	6	13.07	5.57	0.55	4	6	6.85	2.95	1.96
Max./Min.		17.20	7.34	0.32	Max./Min.		13.44	5.78	0.51
2	1	11.37	5.76	0.52	5	1	7.63	17.06	0.06
2	3	∞	∞	0.00	5	2	1.52	3.41	1.47
2	4	11.11	5.63	0.54	5	3	1.52	3.41	1.47
2	5	11.93	6.04	0.47	5	4	6.84	15.30	0.07
2	6	13.07	6.62	0.39	5	6	6.85	15.32	0.07
Max./Min.		∞	∞	0.00	Max./Min.		7.63	17.06	0.06
3	1	11.37	12.71	0.11	6	1	11.37	9.61	0.18
3	2	∞	∞	0.00	6	2	3.27	2.77	2.23
3	4	11.11	12.42	0.11	6	3	3.27	2.77	2.23
3	5	11.93	13.33	0.10	6	4	6.84	5.78	0.51
3	6	13.07	14.62	0.08	6	5	13.44	11.36	0.13
Max./Min.		∞	∞	0.00	Max./Min.		13.44	11.36	0.13

Table 5 External reliability for two outliers in levelling network example

i	j	y_{02} (m)	y_{03} (m)	y_{05} (m)
1	2	4.36	1.34	1.53
1	3	4.36	11.90	1.53
1	4	4.05	2.75	0.38
1	5	8.07	2.13	6.92
1	6	7.01	1.34	1.53
2	3	4.02	∞	1.41
2	4	4.83	2.00	1.54
2	5	5.52	1.72	2.55
2	6	6.40	1.34	1.53
3	4	4.83	11.90	1.54
3	5	5.52	12.78	2.55
3	6	6.40	13.85	1.53
4	5	1.74	2.54	5.65
4	6	1.74	2.54	1.19
5	6	1.74	2.54	7.99
Max.		8.07	∞	7.99

on θ being equal to one or two, pass at the 0.1% significance level. However if two outliers are added in observations 1 and 4, of −8.5 and 7 m, respectively, then it is discovered that all of the outlier tests pass as shown in Table 6. Since the critical value is, 10.83 for θ equal to one and 13.82 for θ equal to two. This is despite both outliers being greater than their MDBs, based on θ equal to one, in Table 1. However, the situation can be explained from Table 4 since both of the outliers are less than their MDBs based on θ being equal to two. If the outliers in observations 1 and 4 are increased to −14 and 12 m, respectively, which are greater than their MDBs. Then it can be seen from Table 6 that the outlier tests based on θ being equal to one sill pass, however, the outlier tests based on θ equal to two identifies observations 1 and 4.

Table 6 Outlier testing for two outliers in measurements 1 and 4

$\theta = 1$			$\theta = 2$			
i	w^{2} [a]	w^{2} [b]	i	j	w^{2} [a]	w^{2} [b]
1	3.00	2.98	1	2	4.84	5.69
2	3.81	4.90	1	3	4.84	5.69
3	3.81	4.90	1	4	10.55	28.00
4	0.78	0.06	1	5	4.90	12.35
5	2.15	1.37	1	6	4.84	5.69
6	3.98	4.22	2	3	3.81	4.90
			2	4	3.84	5.27
			2	5	4.13	4.90
			2	6	4.84	5.69
			3	4	3.84	5.27
			3	5	4.13	4.90
			3	6	4.84	5.69
			4	5	10.45	23.52
			4	6	10.45	23.52
			5	6	10.45	23.52

[a] Two outliers, of −8.5 m in observation 1 and 7 m in observation 4
[b] Two outliers, of −14 m in observation 1 and 12 m in observation 4

As another example, consider two outliers in observations 2 and 3 of −50 and 50 m, respectively. If outlier testing is carried out for θ equal to one or two, then all of the outlier tests pass at the 0.1% significance level as shown in Table 7. In addition, if the outliers are increased to −500 and 500 m then if is found that all of the outlier tests sill pass. The reason for this can be explained from Table 4 since there is no reliability if observations 2 and 3 are considered outliers.

6 Concluding remarks

It is often assumed that there is at most a single outlier present within a set of measurements. However, multiple outliers are possible. Consequently, measures of reliability have been generalised for multiple outliers based on the global model test and the multiple outlier statistic.

Existing measures of reliability have been generalised to multiple outliers and where necessary additional measures have been developed. The additional measures developed include, MDBs, controllability numbers and reliability numbers. The derivation is based on the application of the

Table 7 Outlier testing for two outliers in measurements 2 and 3

$\theta = 1$			$\theta = 2$			
i	$w^{2\,a}$	$w^{2\,b}$	i	j	$w^{2\,a}$	$w^{2\,b}$
1	0.40	0.40	1	2	1.30	1.30
2	1.26	1.26	1	3	1.30	1.30
3	1.26	1.26	1	4	0.57	0.57
4	0.52	0.52	1	5	1.35	1.35
5	0.63	0.63	1	6	1.30	1.30
6	0.69	0.69	2	3	1.26	1.27
			2	4	1.38	1.38
			2	5	1.34	1.34
			2	6	1.30	1.30
			3	4	1.38	1.38
			3	5	1.34	1.34
			3	6	1.30	1.30
			4	5	0.71	0.71
			4	6	0.71	0.71
			5	6	0.71	0.71

[a] Two outliers, of −50 m in observation 2 and 50 m in observation 3
[b] Two outliers, of −500 m in observation 2 and 500 m in observation 3

Rayleigh–Ritz Theorem, and the concept of the multiple correlation coefficient.

It has been shown that internal reliability measures for multiple outliers are equal to or poorer than their corresponding values for a single outlier. The degree to which internal reliability measures are degraded is based on the multiple correlation coefficients, with small correlations desired in order to provide optimum reliability. In addition, it was shown that the external reliability values are larger when multiple outliers are considered. Hence, lower levels of internal and external reliability are achieved when multiple outliers are considered.

While how to determine the number of outliers, existing in a data set, is still open. The results also highlight the limitations of fixing the number of outliers to be considered in a geodetic network. If a network is designed to be reliable against one outlier, but the actual network contains more. Then there is a potential for the network to be significantly less reliable than what it is believed. Hence, this may lead to distortions existing in networks that are considered reliable. If the number of outliers considered in the design is set such that the probability of additional outliers is remote. Then it is highly unlikely that the network will contain distortions, and therefore can be safely considered reliable.

Appendix A

The Rayleigh–Ritz Theorem, also known as Rayleigh quotient, states that for a given symmetrical matrix $\mathbf{C}$, and a symmetrical positive definite matrix $\mathbf{B}$, that are of the same order, with the random vector $\mathbf{x}$ are bound according to Barrett (2007),

$$\lambda_{\mathrm{Min}} \leq \frac{\mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x}} \leq \lambda_{\mathrm{Max}} \tag{69}$$

where λ_{Min} and λ_{Max} are the minimum and maximum eigenvalues, respectively, of the general eigenvalue problem,

$$\mathbf{C}\mathbf{u} = \lambda\mathbf{B}\mathbf{u} \tag{70}$$

The random vector $\mathbf{x}_{\mathrm{Max}}$ that maximises Eq. (69) can also be obtained from the eigenvector corresponding to λ_{Max} as,

$$\mathbf{x}_{\mathrm{Max}} = \mathbf{u}_{\mathrm{Max}} \tag{71}$$

and, similarly $\mathbf{x}_{\mathrm{Min}}$ that minimises Eq. (69) can also be obtained from,

$$\mathbf{x}_{\mathrm{Min}} = \mathbf{u}_{\mathrm{Min}} \tag{72}$$

The general eigenvalue problem in Eq. (70) can be simplified to the normal eigenvalue problem by either multiplying through by $\mathbf{B}^{-1}$ to give,

$$(\mathbf{B}^{-1}\mathbf{C})\mathbf{u} = \lambda\mathbf{u} \tag{73}$$

or alternatively by making the substitution,

$$\mathbf{u} = \mathbf{U}^{-1}\mathbf{u}_{*} \tag{74}$$

where $\mathbf{U}$ is the upper triangle of the Cholesky decomposition of $\mathbf{B}$, and then multiplying through by $(\mathbf{U}^{\mathrm{T}})^{-1}$ to give,

$$\left(\mathbf{U}^{\mathrm{T}}\right)^{-1}\mathbf{C}\mathbf{U}^{-1}\mathbf{u}_{*} = \lambda\mathbf{u}_{*} \tag{75}$$

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