Single particle motion

Plasma is a collection of a very large number of charged particles moving in, and giving rise to, electromagnetic fields.

Before going to the statistical descriptions, let us learn about the motion of individual particles in given EM-fields:

- Guiding centre approximation
- Adiabatic invariants
- Motion in a dipole field
- Motion in the field of a current sheet

Guiding centre approximation

Equation of motion of a charged particle is

\[ \frac{dp}{dt} = q(E + v \times B) + F_{\text{non-EM}} \]

(assume, for the time being, nonrelativistic motion; \( \gamma = 1 \) and \( p = mv \))

The case of \( E = \text{const} \) and \( B = 0 \) (neglect the non-EM forces) is trivial:

\[ \frac{dE}{dt} = 0 , \text{ i.e., linear acceleration in the direction of } E \]

Consider next the case \( E = 0 \) and \( B = \text{const} \)

The equation of motion is now

\[ m \frac{dE}{dt} = q(v \times B) \]

\[ \Rightarrow m \frac{dE}{dt} \cdot v = \frac{d}{dt} \left( \frac{mv^2}{2} \right) = 0 \]

Consequently, the kinetic energy and the speed remain constant
The orbit can be found in several ways, let’s do it very simply starting from the components of the equation of motion (let \( B \) be in the \( z \)-direction):

\[
\begin{align*}
qv_x &= qBv_y \\
-qBv_x &= -\omega_L^2 v_x \\
0 &= -\omega_L^2 v_y
\end{align*}
\]

This describes helical motion that
- is constant in the direction of \( B \)
- circular in the \( xy \) plane

The radius of the circle is (Larmor radius)

\[
r_L = \frac{v_\perp}{|\omega_L|} = \frac{mv_\perp}{|qB|}.
\]

The gyro period (cyclotron period, Larmor time) is:

\[
\tau_L = \frac{2\pi}{|\omega_L|}.
\]

The centre of the gyro motion is called guiding centre (GS)

electron

An electron rotates around its guiding centre in the right-hand sense wrt. the magnetic field!

The pitch angle (\( \alpha \)) of the helical path is defined by

\[
\tan \alpha = \frac{v_\perp}{v_\parallel}.
\]

\[
\begin{align*}
\alpha &= \arcsin\left(\frac{v_\perp}{v}\right) - \arccos\left(\frac{v_\parallel}{v}\right) \\
\alpha &= 90^\circ \rightarrow v = v_\perp \\
\alpha &= 0^\circ \rightarrow v = v_\parallel
\end{align*}
\]

Note that the pitch angle is the complementary angle to the ordinary “pitch of a screw.”

In a uniform magnetic field \( \alpha \) is constant

In a non-uniform magnetic field \( \alpha \) and the ratio between parallel and perpendicular velocities changes

The frame of reference where \( v_\parallel = 0 \) : Guiding centre system (GCS)

Decomposition of the motion to the motion of the guiding centre and to the gyro motion is called the guiding centre approximation

Hannes Alfvén: Guiding centre approximation is valid in temporally and spatially varying fields when variations are small during one gyroperiod \( \rightarrow \) perturbation theory
The frame of reference where \( \mathbf{v} \parallel 0 \) : Guiding centre system (GCS)

Decomposition of the motion to the motion of the guiding centre and to the gyro motion is called the \textbf{guiding centre approximation}

In the GCS the charge causes an electric current: \( I = q / \tau_L \)

The \textbf{magnetic moment} associated with the circular loop is

\[
\mu = I \pi r_L^2 = \frac{q^2 r_L^2 B}{2m} = \frac{m v_L^2}{2B} = \frac{W}{B}
\]

or, in the vector form \( \mu = \frac{1}{2} q r_L \times v_L \)

Clearly: \( \mu \) is always opposite to \( \mathbf{B} \) (\( r_L \) depends on the sign of \( q \))

Thus plasma can be considered a \textbf{diamagnetic medium}:

Placing a large number of charged particles to external magnetic field reduces the field.

---

\[ \textbf{E} \times \mathbf{B} \text{ drift} \]

Let \( \mathbf{E} = \text{const} \) and \( \mathbf{B} = \text{const} \)

The eq. of motion along \( \mathbf{B} \) is \( m \ddot{v}_1 = qE \)

\( \rightarrow \) constant acceleration parallel/antiparallel to \( \mathbf{B} \)

\( \rightarrow \) very rapid cancellation of large-scale \( \mathbf{E} \) in plasma!

The perpendicular components of the eq. of motion are

\[
\begin{align*}
\dot{v}_x &= \omega_0 v_y = \frac{q}{m} E_z \\
\dot{v}_y &= -\omega_0 v_x \\
\Rightarrow \quad \ddot{v}_x &= -\omega_0^2 v_x \\
\ddot{v}_y &= -\omega_0^2 (v_y + E_z/B)
\end{align*}
\]

Substitution \( v'_y = v_y + E_z/B \) leads again to gyro motion but now the GC drifts in the \( y \)-direction with speed \( E_z/B \)

In vector form:

\[ \mathbf{\dot{v}} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \]

Same as making Lorentz transformation to GCS:

\[ \mathbf{E}' = \mathbf{E} + \mathbf{w} \times \mathbf{B} \quad \text{(non-relat. } \gamma = 1) \]

where \( \mathbf{E}' = 0, \mathbf{E} = -\mathbf{w} \times \mathbf{B} \)

All charged particles drift to the same direction \( \mathbf{\perp E} \) and \( \mathbf{\perp B} \)
Other non-magnetic drifts

Write the perpendicular eq. of motion in the form
\[ \frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m} (\mathbf{v}_\perp \times \mathbf{B}) + \mathbf{F}_\perp/m. \]

Assume that \( \mathbf{F}_\perp \) gives rise to a drift \( \mathbf{v}_D \), and transform \( \mathbf{v}_\perp = \mathbf{v}_\perp' + \mathbf{v}_D \).

\[ \Rightarrow \frac{d\mathbf{v}_\perp'}{dt} = \frac{q}{m} (\mathbf{v}_\perp' \times \mathbf{B}) + \frac{q}{m} (\mathbf{v}_D \times \mathbf{B}) + \frac{\mathbf{F}_\perp}{m}. \]

In GCS the last two terms must sum to 0 \( \Rightarrow \mathbf{v}_D = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2} \) (*).

This requires \( F/qB \ll c \). If \( F > qcB \), the GC approximation cannot be used!

Inserting \( \mathbf{F}_\perp = q\mathbf{E} \) into (*) we get the ExB-drift

\[ \mathbf{F}_\perp = mg \text{ gives the gravitational drift} \quad \mathbf{v}_g = \frac{mg \times \mathbf{B}}{qB^2} \propto \frac{m}{q}. \]

Separates charges \( \rightarrow \) current

Slow time variations in \( \mathbf{E} \rightarrow \) polarization drift

The corresponding polarization current is

\[ \mathbf{J}_P = n_e (\mathbf{v}_P - \mathbf{v}_e) = \frac{n_e (m_i + m_e)}{B^2} \frac{d\mathbf{E}_\perp}{dt} \propto \frac{n_e m_i}{B^2} \frac{d\mathbf{E}_\perp}{dt} \text{ carried by ions!} \]

Drifts due to non-uniform magnetic field

Assume static but inhomogeneous magnetic field. Guiding centre approximation is useful if the gradients of \( \mathbf{B} \) are small as compared to the gyro motion:

\[ |\nabla B|_\perp \ll B/v_T, \quad |\nabla B|_\parallel \ll (\omega_c/v_\parallel)B. \]

Expand the field as a Taylor series about the GC:

\[ \mathbf{B}(r) \approx \mathbf{B}_0 + r \cdot (\nabla \mathbf{B})_0 + \ldots \text{ where } \mathbf{B}_0 \text{ is the field at GC} \]

Straightforward (but a tedious) calculation yields the force \( \mathbf{F} = - \mu \nabla B \)

Along \( \mathbf{B} \): acceleration

\[ \frac{d\mathbf{v}_\parallel}{dt} = -\frac{B}{m} \nabla \mathbf{B} \]
Perpendicular to $\mathbf{B}$: gradient drift

Use $v_D = \frac{F_\perp \times \mathbf{B}}{qB^2}$ \implies $v_D = \frac{\mu}{qB^2} \mathbf{B} \times (\nabla B) = \frac{W_\perp}{qB^3} \mathbf{B} \times (\nabla B)$

Note: the gradient drift separates different particle species → current

Inhomogeneous magnetic field is usually also curved. The charged particle experiences the centrifugal force:

$$F_C = -\frac{mv_G^2}{R_C^2}$$

$R_C$ is the radius of curvature (positive inwards) and $w_G$ is the velocity of the GC along $\mathbf{B}$ (not exactly $v_G$ but $w_G \approx v_G$ is for us a good enough approximation)

Applying the force $F_C = -\frac{mv_G^2}{R_C^2}$ as before we get the curvature drift

$$v_C = -\frac{mv_G^2}{q} R_C \times \mathbf{B}$$

$$\frac{R_C}{R_C} = \frac{(\mathbf{B} \cdot \nabla \mathbf{B})_\perp}{B^2} \quad \text{exercise} \quad v_C = \frac{mv_G^2}{qB^3} \mathbf{B} \times (\mathbf{B} \cdot \nabla \mathbf{B})$$

If there are no local currents ($\nabla \times \mathbf{B} = 0$), $v_C = \frac{mv_G^2}{qB^3} \mathbf{B} \times \nabla \mathbf{B}$

and $v_G$ and $v_C$ can be combined to

$$v_{GC} = \frac{W_\perp + 2W_\parallel}{qB^3} \mathbf{B} \times \nabla \mathbf{B} = \frac{W}{qBR_C} (1 + \cos^2 \alpha) \mathbf{n} \times \mathbf{t} \quad \text{unit vectors} \quad \mathbf{n} \parallel \mathbf{R}_C, \mathbf{t} \parallel \mathbf{B}$$

These are first order drifts. The same procedure can be continued to higher orders:
- determine the force due to the 1st order drift
- calculate the 2nd order drift speed using the same formula (+) as above
Particle drifts in dipole geometry

\[ \mathbf{v}_C = \frac{m_0^2}{qB^2} B \times \nabla B \]

Positive charges drift to the west,
Negative charges to the east
→ net westward current
around the Earth

Next we will learn about the mirror effect

Adiabatic invariants

Symmetry principles: periodic motion ↔ conserved quantity
symmetry ↔ conservation law

What if the motion is almost periodic?

Hamiltonian formulation:
Let \( q \) & \( p \) be canonical variables and the motion be almost periodic \( \Rightarrow \)

\[ I = \oint p dq \]

is constant, called adiabatic invariant

Example: Consider a charged particle in Larmor motion.
Assume that the \( B \) does not change much during within one circle.
The canonical coordinate is \( r_L \) and the canonical momentum \( p = m \mathbf{v} + q \mathbf{A} \)

\[ I = \oint p_L \cdot d\mathbf{r}_L = \oint m\mathbf{v}_L \cdot d\mathbf{r}_L + q \oint (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \]

\[ = \int_0^{2\pi r_L} m\mathbf{v}_L \cdot dl + q \oint \mathbf{B} \cdot d\mathbf{S} \]

\[ = 2\pi mv_L r_L B - q |B| r_L^2 \]

i.e., the magnetic moment
is an adiabatic invariant
Example: Lorentz-Einstein pendulum

Assume that the length of the pendulum is changed slowly (as compared to $\omega$) $\Rightarrow$

$\omega = \sqrt{\frac{g}{l}}$ changes slowly

Now the energy per mass unit $W = \frac{1}{2}l^2 \dot{\theta}^2 + \frac{1}{2} g l \dot{\theta}^2$ is not conserved.

Lorentz asked Einstein at a conference in 1911: What is the conserved quantity in this case?

Einstein: $W/\omega$

This was a relevant question while quantum mechanics was being developed. Recall that

$W/\omega = \hbar = \text{const}$

Relation to the conservation of magnetic moment:

$\mu = \frac{W_\perp}{B} = \frac{q}{m} \frac{W_\perp}{\omega_\perp}$

First adiabatic invariant

In plasma physics $\mu$ is called the first adiabatic invariant. Let’s show its conservation by direct calculation in the case of static weakly inhomogeneous $B$ (GC approximation)

The system is conservative, i.e., total energy is conserved:

$W = W_\parallel + W_\perp = \text{const} \Rightarrow \frac{dW_\parallel}{dt} + \frac{dW_\perp}{dt} = 0$

$W_\perp = \mu B \Rightarrow \frac{dW_\perp}{dt} = \mu \frac{dB}{dt} + \frac{d\mu}{dt} B$ along the path of GC

Change in parallel energy:

$m \frac{d\nabla\parallel}{dt} = -\mu \nabla\parallel B = -\mu \frac{dB}{ds}$ multiply LHS by $\nabla\parallel$ and RHS by $ds/dt \Rightarrow$

$\frac{dW_\parallel}{dt} = -\mu \frac{dB}{dt}$ and thus $\frac{dW_\parallel}{dt} + \frac{dW_\perp}{dt} = B \frac{d\mu}{dt} = 0$

i.e., $\mu$ is constant in the guiding centre approximation
Assume next that $\mathbf{B}$ is time-dependent but changes slowly: $\partial B/\partial t \ll \omega_c$.

Now the electric field must be taken into account: $\partial \mathbf{B}/\partial t = -\nabla \times \mathbf{E}$.

The electric field changes the perpendicular energy:

$$\frac{dW_\perp}{dt} = q(E \cdot v_\perp)$$

which during one gyro period yields the work:

$$\Delta W_\perp = q \int_0^{2\pi/\omega_c} E \cdot v_\perp dt$$

As the field changes slowly, replace the time integral by contour integral:

$$\Delta W_\perp = q \oint_C \mathbf{E} \cdot d\mathbf{l} = q \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -q \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

For small changes of the field:

$$\frac{\partial \mathbf{B}}{\partial t} \rightarrow \omega_c \Delta B/2\pi \quad \Rightarrow \quad \Delta W_\perp = \frac{1}{2} q \omega_c B_\perp^2 \Delta B = \mu \Delta B$$

Because:

$$\Delta W_\perp = \mu \Delta B + B \Delta \mu \quad \mu \text{ must again remain constant}$$

Note that also the flux enclosed by the gyro orbit $\Phi = B\pi r_L^2$ is constant:

$$\Phi = \frac{2\pi mA}{q}$$

### Magnetic mirror

In guiding centre approximation both $W$ and $\mu$ are conserved.

If $B$ increases slowly, $W_\parallel$ increases slowly, thus $W_\perp$ decreases.

What happens when $W_\parallel \rightarrow 0$?

$$\mu = \frac{mv^2 \sin^2 \alpha}{2B}$$

As $\mu$ and $v^2$ are conserved, $\alpha$ and $B$ are related through:

$$\sin^2 \alpha_1 = \frac{B_1}{B_2}$$

When $\alpha \rightarrow \pi/2$, the force $\mathbf{F} = -\mu \nabla_\parallel B$ on the GC turns the charge back (mirror force) and the mirror field $B_m$ for a charge that at $B_0$ has the pitch-angle $\alpha$ is given by:

$$\sin^2 \alpha_0 = B_0/B_m$$

If $B_2 > B_1$:

$$\frac{W_\perp}{W_\parallel} = \frac{B_2}{B_1} \quad \Rightarrow \quad W_\parallel > W_\perp$$

adiabatic heating
Magnetic bottle

A simple magnetic bottle consists of two mirrors facing each other. A charged particle is trapped in the bottle if

$$\arcsin \sqrt{\frac{B_0}{B_m}} \leq \alpha_0 \leq 180^\circ - \arcsin \sqrt{\frac{B_0}{B_m}}$$

Otherwise it is said to be in the loss-cone and escape at the end of the bottle.

There are much more complicated trapping configurations, e.g. stellarator:

The dipole field of the Earth is a large magnetic bottle.

Second adiabatic invariant

Consider the bounce period of a charge in a magnetic bottle

$$\tau_b = 2 \int_{v_m}^{v_m} \frac{ds}{v(s)} = \frac{2}{v} \int_{s_m}^{s_m} \frac{ds}{(1 - B(s)/B_m)^{1/2}}$$

If \( \tau_b \frac{dB/dt}{B} \ll 1 \) then \( J = \int p v ds \) is constant (second adiabatic invariant).

If the mirror points move toward each other, \( p \) and thus also \( W \) increases (c.f. a tennis racket hitting a ball). This is known as Fermi acceleration. It was an early proposal by Enrico Fermi to explain acceleration of galactic cosmic rays. Today shock acceleration is understood to be more important (see Space applications of plasma physics, period II; Advanced space physics, period III).

Enrico Fermi
Third adiabatic invariant

If the perpendicular drift of the GC is nearly periodic (e.g. in a dipole field), the magnetic flux through the GC orbit

\[ \Phi = \int A \cdot d\mathbf{x} \]

is conserved.

This is the third adiabatic invariant.

Summary of adiabatic invariants

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<td>( \tau \gg \tau_L )</td>
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<tr>
<td>longitudinal</td>
<td>longitudinal velocity of GC ( \mathbf{w}_\parallel )</td>
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<tr>
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<td>( \tau \gg \tau_d \gg \tau_b \gg \tau_L ) and ( \mu ) and ( J ) const</td>
</tr>
</tbody>
</table>

There is a specific energization mechanism for each invariant

\( \mu : \) \( W_\mu \) changed by changing the Larmor radius (i.e., \( |B| \))

\( J : \) \( W_J \) changed by stretching or shortening the magnetic bottle

\( \phi : \) \( W \) changed by compressing or expanding the drift surface
Motion in a dipole field

The dipole field is familiar from basic electrodynamics.

We use the notations of geomagnetism:
- the latitude \( \lambda \): zero at equator, increases towards north
- the longitude \( \phi \) increases towards east
- instead of dipole moment \( M_E \) (for the Earth), we use \( k_0 = \mu_0 M_E/4\pi \) (also often called dipole moment)

\[
M_E = 8 \times 10^{22} \text{ Am}^2
\]

\[
k_0 = \frac{M_E}{4 \pi} \quad \text{(SI: Wb = Tm²)}
\]

\[
= 8 \times 10^{18} \text{ Wb m} \quad \text{(Gaussian units: T = 10^-4 T)}
\]

\[
= 0.3 \times 10^{-3} \text{ GRe}^2 
\]

\( \text{(Re \approx 6370 km, Earth’s radius)} \)

As the dipole field is curl-free, it can be given as \( \nabla \Psi \), where

\[
\Psi = -k_0 \cdot \nabla \frac{1}{r} = -k_0 \frac{\sin \lambda}{r^2}
\]

\[
\Rightarrow \quad B = \frac{1}{r^2} \left[ 3(k_0 \cdot \hat{r}) \hat{r} - k_0 \right]
\]

components of \( B \):

- \( B_r = -\frac{2k_0}{r^3} \sin \lambda \)
- \( B_\lambda = \frac{k_0}{r^3} \cos \lambda \)
- \( B_\phi = 0 \) \( \Leftarrow \) (azimuthal symmetry)

The field lines are found from the equation

\[
\frac{dr}{B_r} = \frac{d\lambda}{B_\lambda}
\]

\[
\Rightarrow \quad r = r_0 \cos^2 \lambda
\]

The length of the line element is

\[
ds = \left( \frac{dr^2 + r^2 d\lambda^2}{(1 + 3 \sin^2 \lambda)^{1/2}} \right)^{1/2} = r_0 \cos \lambda (1 + 3 \sin^2 \lambda)^{1/2} d\lambda
\]

Note the important geometrical factor \( (1 + 3 \sin^2 \lambda)^{1/2} = (4 - 3 \cos^2 \lambda)^{1/2} \)

Every field line is determined by two parameters:
- (constant) longitude \( \phi_0 \)
- distance \( r_0 \) where the line crosses the equator

Define the L-parameter:

\[
L = r_0 R_E
\]

\( R_E \) is the radius of Earth (or another planet)
For given $L$ the field line crosses the surface at latitude $\lambda_c = \arccos \frac{1}{\sqrt{L}}$

$B$ on a given field line as a function of latitude:

$$B(\lambda) = \left[ B_r(\lambda)^2 + B_\lambda(\lambda)^2 \right]^{1/2} = \frac{k_0 (1 + 3 \sin^2 \lambda)^{1/2}}{r_0^3 \cos^6 \lambda}$$

for the Earth:

$$\frac{k_0}{r_0^3} = 0.3 \frac{L}{3} G = 3 \times 10^{-5} \frac{L}{3} T$$

In calculations of particle orbits we need the curvature radius of $B$, $R_C$.

$$1/R_C = |\nabla \times B|/|B|$$

$$\Rightarrow R_C(\lambda) = \frac{r_0}{3} \cos \lambda \frac{(1 + 3 \sin^2 \lambda)^{3/2}}{2 - \cos^2 \lambda}$$

Guiding centre approximation is applicable in dipole field, if $R_L \ll R_C$ (*).

Expressed in terms of the rigidity of the particle $mv/|q|$

$$r_L \frac{\nabla \times B}{|B|} = \frac{mv_u}{|q|r_0 B} \times \frac{mv_u}{|q|r_0 B}\text{In cosmic ray physics rigidly is often defined by } R = pc/|q|, \text{ and thus } [R] = V$$

Condition (*) reduces to $ \frac{mv_u}{|q|} \ll r_0 B$

The dipole field is a magnetic bottle. Let $\lambda_m$ be the mirror latitude. The pitch-angle at equator is determined by

$$\sin^2 \alpha_0 = \frac{B_0}{B(\lambda_m)} = \frac{\cos^6 \lambda_m}{(1 + 3 \sin^2 \lambda_m)^{1/2}}$$

The width of the loss cone at equator is given by

$$\sin^2 \alpha_{ul} = L^{-3}(4 - 3/L)^{-1/2} - (4L^6 - 3L^5)^{-1/2}$$

The particle leaks out if its $\alpha_0 < \alpha_{ul}$

At the Earth most particles are lost at an altitude of about 100 km (in the ionosphere)
Consider the bounce motion in the dipole field:

\[ \tau_b = 4 \int_0^{\lambda_0} \frac{d\lambda}{v_{\perp}} = 4 \int_0^{\lambda_0} \frac{d\lambda}{d\lambda} \frac{d\lambda}{v_{\perp}} \quad \Rightarrow \quad \tau_b = 4r_0 \int_0^{\lambda_0} \cos(\lambda(1 + 3 \sin^2 \alpha) \lambda)^{1/2} d\lambda = \frac{4r_0}{v} f(\alpha_0) \]

For pitch-angles \(30^\circ \leq \alpha_0 \leq 90^\circ\): \(f(\alpha_0) \approx 1.30 - 0.56 \sin^2 \alpha_0\)

A useful order-of-magnitude estimate: \(\tau_b \approx \frac{4r_0}{v} \approx 0.085 \cdot \frac{L}{B} \quad \Rightarrow \beta = v/c\)

In the Earth’s dipole field 1-keV electrons bounce in seconds, 1 keV protons in minutes.

As \(\nabla \times \mathbf{B} = 0\)

\[ v_{\text{GC}} = \frac{W}{qBR_C} (1 + \cos^2 \alpha(\lambda)) \]

\[ = \frac{3m^2 \gamma_0^2 \cos^5 \lambda(1 + \sin^2 \lambda)}{2 q k_0} (1 + 3 \sin^2 \lambda)^{1/2} \left[ 2 - \sin^2 \alpha_0 (1 + 3 \sin^2 \lambda)^{1/2} \right] \cos^6 \lambda \]

Angular drift velocity depends on the latitude

Often the useful quantity is the average angular drift velocity:

\[ \langle \dot{\phi} \rangle = \frac{1}{\tau_b} \int_{0}^{\lambda_0} \frac{d\phi}{d\lambda} \, d\lambda = \frac{1}{\tau_b} \int_{0}^{\lambda_0} \frac{d\phi}{\cos \alpha} \, d\lambda = \frac{3 \alpha_0^2 \gamma_0}{2 q k_0} g(\alpha_0) \]

For \(30^\circ \leq \alpha_0 \leq 90^\circ\): \(g(\alpha_0) \approx 0.7 + 0.3 \sin(\alpha_0)\)

For particles staying on equator \(\alpha_0 = \pi/2\) \(\langle \dot{\phi}_0 \rangle = \frac{3m^2 \gamma_0 \mathbf{E} \mathbf{L}}{2 q k_0}\)

For relativistic particles this is \(\langle \dot{\phi}_0 \rangle = \frac{3m^2 \gamma_0 \mathbf{E} \mathbf{L}}{2 q k_0} \gamma \gamma^2 = (1 - \beta^2)^{-1/2}\)

Finally the average drift time is given by \(\tau_d = \frac{2\pi}{\langle \dot{\phi} \rangle} \approx 1.0 \cdot 10^4 \frac{m_0 |q|}{B eL \gamma^2 g(\alpha_0)} \) s.
Van Allen belts (or radiation belts)

Typical energies:
- Inner belt protons: 0.1 MeV – 40 MeV
- Outer belt electrons: keV – MeV

Typical drift periods
- keV particles: 100s of hours
- MeV particles: 10s of minutes

Recall:
- $m_p = 938 \text{ MeV}/c^2$
- $m_e = 511 \text{ keV}/c^2$

Ions are nonrelativistic, the most energetic electrons are relativistic.

Current sheets

Current sheets are important in plasma physics. They separate different plasma domains and they are sites of the most important energy release process, magnetic reconnection (see e.g., the lectures on Space applications of plasma physics (period II) or Advanced space physics (spring 2012)).

The Harris model is a 2D example:

$$\mathbf{B} = B_0 \text{tanh} \left( \frac{x}{L} \right) \mathbf{e}_x + B_n \mathbf{e}_z$$

$B_n$ and $B_0$ are constant and $B_n \ll B_0$

The current is according to Ampère’s law

$$J_y = \frac{B_0}{\mu_0 L} \nabla \times A \left( \frac{y}{L} \right)$$

One-dimensional and two-dimensional Harris models
Motion in the current sheet

Examples of orbits of positive charges in and around a current sheet

When a dipole field is stretched, a current-sheet is introduced. For particles, whose $R_L \rightarrow R_C$ across the current sheet, regular motion becomes non-adiabatic, i.e., $\mu$ is no more conserved.

An example of transition to chaos